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Bianchi I and V cosmologies with Hu-Sawicki $f(R)$ gravity in Palatini formalism

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¹ Author to whom any correspondence should be addressed.E-mail: debikabanik05@gmail.com, sebikabanik063@gmail.com and kalyanbhuyan@gmail.com**Keywords:** Bianchi type-I metric, Bianchi type-V metric, $f(R)$ gravity, Palatini formalism, dynamical systems approach

Abstract

The main objective of this work is the pursuit of a detailed analysis of the cosmological solutions corresponding to each fixed point belonging to an exclusive form of $f(R)$ initially introduced by Hu and Sawicki in the background of Palatini $f(R)$ gravity. Fixed points and their associated solutions for the anisotropic Bianchi type I and V models are derived by utilizing an approach used to solve autonomous ordinary differential equations. Building on this background, an exhaustive study on the phase space is accomplished and the existence of a sequence of evolution that starts from unstable anisotropic universe to a stable phase of late times de Sitter expansion through the standard radiation and matter dominated era has been established.

1. Introduction:

In recent years, modified theories of gravity have earned great interest among researchers all over the globe as these theories are capable of reproducing the accelerated phase of the late universe. Although many theories of modified gravity have been constructed to explain such late time acceleration, $f(R)$ gravity [1–10] is considered one of the most extensively studied and as the simplest of such theories. In $f(R)$ theories, the Ricci Scalar R in the Lagrangian density corresponding to the standard Einstein-Hilbert action is replaced by some arbitrary function of R . The field equations corresponding to the modified action can be obtained executing different formalisms of $f(R)$ gravity and in this analysis we pay our attention to the Palatini formalism. In this formalism, field equations are obtained by varying the action corresponding to the metric as well as the connection, considering them as two independent variables. An additional assumption in Palatini formalism is that the matter action is independent of the connection. In addition to the fact that the field equations correspond to mathematically simple second order differential equations, the Palatini formalism has gained sufficient interest from the perspective of the cosmological tests [11]. Palatini formalism of $f(R)$ gravity is capable of reproducing the three post inflationary epochs of the standard cosmology for some of the cosmological models based on a power law functional form, which is not possible in case of the standard metric formalism. In this analysis, we study an exclusive form of $f(R)$, as originally proposed by Hu and Sawicki [11–18] within the Palatini approach. The cosmological consequences of Palatini $f(R)$ gravity for an exclusive form of $f(R)$ initially introduced by Hu and Sawicki in the background of Friedmann-Leimaitre-Robertson-Walker (FLRW) model has been investigated in [11]. It has been observed that the Hu-Sawicki model is indistinguishable from the standard Λ CDM scenario and is able to accomplish the sequence of radiation dominated, matter dominated and accelerated expanding period in the Palatini formalism. Hu and Sawicki have provided the metric variation for a class of $f(R)$ gravity which accelerates the expansion without considering a cosmological constant and also satisfies the cosmological as well as solar system tests corresponding to the small field limit of the parameter space [12]. A detailed analysis of the dynamics corresponding to Hu-Sawicki $f(R)$ model is presented in [13]. A full phase space analysis has been performed for the FLRW metric and several de Sitter fixed points have been identified along with a matter like unstable fixed point for a particular value of the model parameters. Authors of [14] have focused on the analysis of the viable Hu-Sawicki $f(R)$ model, which can reproduce the late time accelerated expansion era and also can recover the result that is consistent with the General Relativity (GR) on local scales. A combined investigation on the theoretical as well as observational constraints which

impart some important information about the viable parameter space of this explicit form of $f(R)$ model have been provided. Authors of [15] have explored the two viable $f(R)$ models (Hu-Sawicki and Nojiri-Odinstov), by investigating their corresponding cosmological evolution and studying their scalar-tensor representation. They study the scalar-tensor equivalence of both the viable $f(R)$ models, which provides a very useful tool to extract the necessary information about the possibility of occurrence of cosmological singularities.

Despite being mathematically simpler in comparison to the other formalisms of $f(R)$ gravity, the system of field equations corresponding to Palatini $f(R)$ gravity are non linear in nature and to study the behaviour of the non linear system we use a suitable technique known as Dynamical System Approach (DSA) [19–23]. Such a technique enables us to study the behaviour of the equilibrium points corresponding to the equations of the dynamical system. Fay *et al* have studied the late time acceleration for FLRW model in Palatini $f(R)$ gravity adopting DSA [5].

The acceleration phase of the late universe can be studied in the background of Bianchi type I and V models which are the simple generalization of the FLRW model. FLRW model is spatially homogeneous and isotropic [5, 24–26] whereas Bianchi type I and V models are spatially homogeneous but anisotropic [27–30]. The simplest representation of isotropic and spatially homogeneous stage of present universe can be excellently analysed in the background of FLRW model. However, the early stage of the Universe, which could have been anisotropic [31–37], cannot be explained in the background of this isotropic model. It may be noted that the Cosmic Background Explorer (COBE) confirmed the existence of anisotropy in 1992 [31] and for this discovery the Nobel Prize in Physics in 2006 is awarded to J C Marther and G F Smoot. In addition, some recent probes and experiments, for example, Wilkinson Microwave Anisotropy Probe (WMAP) [32–35] and Planks results [36, 37] also have confirmed the presence of small anisotropies. In a recent paper [38], the authors have used two methods, namely the hemisphere comparison (HC) method and dipole fitting (DF) method in probing the possible preferred direction in the distribution of type Ia supernovae (SNIa). They used these methods in order to test cosmic anisotropy with supernovae data. Evidences of cosmic anisotropy are still accumulating all over the globe. The existence of cosmic anisotropy cannot be fully investigated in the background of isotropic FLRW cosmology. This provides a strong motivation to look for some alternative models to explain such anisotropy. Bianchi models [39–53] pave a way for the researchers to describe the anisotropic phase of early universe and the acceleration phase of late universe.

Our analysis has been carried out for an exclusive form of $f(R)$, as originally proposed by Hu and Sawicki, within the Palatini approach for anisotropic Bianchi I and V models. The main motivation to use the Palatini formalism in the Hu-Sawicki scenario is to check whether such a model is able to reproduce the sequence of cosmological evolution in the presence of anisotropic models or not. Although the FLRW cosmology within the Palatini approach [11] and the anisotropic Bianchi III cosmology within the metric approach [52] in the background of Hu-Sawicki $f(R)$ model has been able show the sequence of the evolution of the Universe, the Palatini version of Hu-Sawicki $f(R)$ gravity has not been yet investigated in presence of anisotropic models. Such anisotropic models are considered as a preferred area of research as it can explain the anisotropy prevailing in the early stage of the Universe.

The content of the present work is outlined as follows. Section 2 presents the field equations of $f(R)$ gravity in the Palatini variational technique. In section 3, we discuss the Hu-Sawicki $f(R)$ model and its cosmological viability. In sections 4 and 5, we derive the fixed points along with their solutions, stabilities and also pursue the phase space analysis for anisotropic Bianchi I and V cosmologies, respectively. Ultimately, section 6 contains the conclusion of the work.

2. The Palatini approach in $f(R)$ gravity

The $f(R)$ gravity action that is defined as the modification of Einstein-Hilbert action is written as

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} f(R) + L_m + L_r \right] \quad (1)$$

where $f(R)$ denotes a function of the Ricci scalar R , g represents the determinant of metric tensor $g_{\mu\nu}$, L_r and L_m are defined as the Lagrangians of matter and radiation respectively, $\kappa^2 = 8\pi G$ and G is the gravitational constant. As discussed in [5], matter and radiation are introduced ad-hoc into the equations of motion.

In Palatini approach, the basic idea is to consider the metric $g_{\mu\nu}$ as well as the affine connection $\Gamma_{\mu\nu}^\lambda$ as two independent variables. The field equations arising due to the variation of the action (7) corresponding to the two independent variables are

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = \kappa T_{\mu\nu} \quad (2)$$

$$\nabla_\lambda(\sqrt{-g}f'(R)g^{\mu\nu}) = 0 \quad (3)$$

where the prime denotes the derivatives with reference to R and ∇ denotes the covariant derivative. $T_{\mu\nu}$ is the tensor of energy momentum defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(L_m + L_r)}{\delta g_{\mu\nu}} \quad (4)$$

3. The Hu-Sawicki $f(R)$ model

In this analysis, particular attention is given to a unique form of $f(R)$ model known as Hu-Sawicki $f(R)$ model [11–15, 52], given by

$$f(R) = R - m^2 \frac{c_1 \left(\frac{R}{m^2}\right)^n}{c_2 \left(\frac{R}{m^2}\right)^n + 1} \quad (5)$$

where c_1, c_2, n are constant parameters and the mass scale is taken as $m^2 = \kappa^2 \rho / 3$. A detailed investigation of the cosmological dynamics corresponding to $n = 1$, in case of Hu-Sawicki $f(R)$ model are presented in [11, 13, 52]. Following [11, 13, 52], we perform our analysis considering $n = 1$, as the value of n is totally unconstrained by the present cosmological data. Thus, for $n = 1$, the above equation reduces to

$$f(R) = R - \frac{2\Lambda}{\frac{\mu^2}{R} + 1} \quad (6)$$

where $\Lambda = m^2 c_1 / 2c_2$ and $\mu^2 = m^2 / c_2$ are two dimensionless parameter, while $m^2 = H_0^2 \Omega_{m0}$ (H_0 and Ω_{m0} are the present-day values of Hubble and matter density parameters, respectively). In the regime $R \ll \mu^2$ the correction to GR is found to be negligible [14]. The above model is found to be practically indistinguishable from the Λ CDM model in the regime $R \gg \mu^2$ [11–14]. Authors of [12] suggest that $R \gg \mu^2$, even in the present epoch. In our analysis, we have considered the condition $R \gg \mu^2$.

A Λ CDM model is the simplest cosmological model which can drive the Universe into the accelerating de Sitter phase. The cosmological constant Λ was originally introduced by Einstein in GR to keep the Universe static. But, now a small positive value of the cosmological constant is considered as a possible candidate for dark energy. The Λ CDM model has provided some well defined predictions that have been able to withstand the rapid as well as continuous advancement of cosmological observational tests. Observations of the large scale structure and recent accelerated expanding universe are in excellent agreement with the Λ CDM model. Despite these major advantages, Λ CDM model has several problems. One of such problems is the cosmological constant problem, which has not been satisfactorily resolved until today. [11, 12, 14] have shown that Hu-Sawicki $f(R)$ model can explain the accelerated expansion without considering a cosmological constant.

Let us now use the scalar-tensor theory of $f(R)$ gravity to examine the sudden singularity for the Hu-Sawicki model.

The action corresponding to the scalar-tensor theory of $f(R)$ gravity is written as

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} (\phi R - V(\phi)) + L_m + L_r \right] \quad (7)$$

where the scalar field ϕ and its potential $V(\phi)$ is given by

$$\phi = f', \quad V(\phi) = Rf' - f \quad (8)$$

In general the explicit expression of the scalar potential in terms of the scalar field for a general n for Hu-Sawicki $f(R)$ model is not possible to achieve. However, this is possible corresponding to the case $n = 1$, such that the scalar potential is given by [14, 15]

$$V(\phi) = \mu^2(1 - \phi) \pm 2\sqrt{2\Lambda\mu^2(1 - \phi)} + 2\Lambda \quad (9)$$

Substituting the value of μ^2 and Λ in the above equation we get

$$V(\phi) = H_0^2 \Omega_{m0}^2 \frac{c_1 + (1 - \phi) \pm 2\sqrt{c_1(1 - \phi)}}{c_2} \quad (10)$$

It is observed that the sudden singularity, where $R \rightarrow \infty$, exists for

$$\phi \rightarrow 1, \quad V \rightarrow H_0^2 \Omega_{m0}^2 \frac{c_1}{c_2} \quad (11)$$

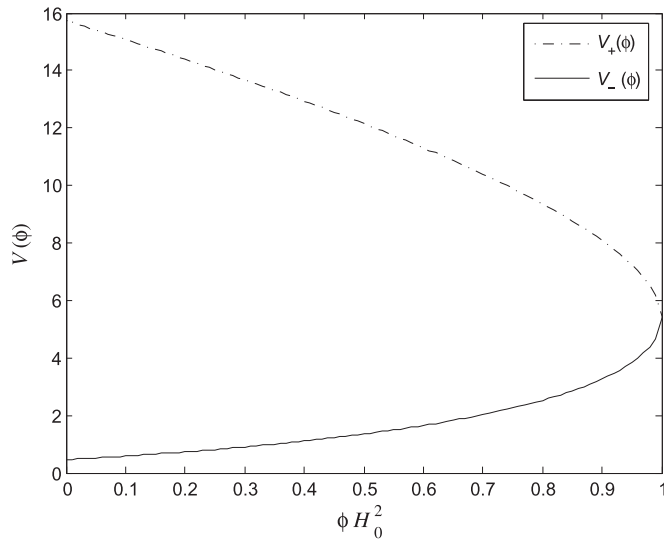


Figure 1. Evolution of the scalar potential ($V_{\pm}(\phi)$) corresponding to Hu-Sawicki model for $n = 1, c_1 = 2, c_2 = 0.1$ and $\Omega_{m0} = 0.27$.

The evolution of scalar potential $V(\phi)$ for $n = 1$ is plotted in figure 1. This figure shows that the two branches of the scalar potential ($V_{\pm}(\phi)$) consist of different asymptotically stable points. While the upper branch ($V_+(\phi)$) stops at the singular point $\phi = 1$, the lower branch ($V_-(\phi)$) accomplished the asymptotically stable de Sitter evolution (see [14]). Therefore in order to avoid the singularity, we have to stay on the lower branch of the scalar potential ($V_-(\phi)$) which leads to the condition

$$V_{(\phi=1)} < H_0^2 \Omega_{m0}^2 \frac{c_1}{c_2} \quad (12)$$

Thus, the singularity can be avoided depending upon the initial conditions and the model parameter values.

4. Bianchi I metric in Palatini $f(R)$ gravity

Our analysis begins with one of the simplest type of anisotropic models known as Bianchi type I. The line element of this anisotropic model is defined as

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)[dy^2 + dz^2] \quad (13)$$

where $A(t)$ and $B(t)$ are considered as the expansion scale factors. x, y, z represent the comoving coordinates whereas t denotes the cosmological time.

The generalised Friedmann field equation in case of the anisotropic Bianchi type I model can be written as

$$\frac{2}{3}f' \left[\theta + \frac{3}{2} \frac{\dot{f}}{f'} \right]^2 - 2f'\sigma^2 - f = \kappa(\rho_m + 2\rho_r) \quad (14)$$

where a dot represents differentiation with respect to cosmological time. θ and σ stands for the volume expansion scalar and the shear scalar, respectively and are defined as

$$\theta = 3\frac{\dot{a}}{a} = \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} \quad (15)$$

$$\sigma^2 = \frac{1}{3} \left[\frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right]^2 \quad (16)$$

where a acts as the average scale factor. ρ_m and ρ_r appear as the matter energy density and radiation energy density, respectively, which satisfy the conservation equations given by

$$\dot{\rho}_m + \theta\rho_m = 0 \quad (17)$$

$$\dot{\rho}_r + \frac{4}{3}\theta\rho_r = 0 \quad (18)$$

The trace free Gauss-Codazzi equation in case of Bianchi type I model for Palatini $f(R)$ gravity is defined as

$$\dot{\sigma} = -\left[\theta + \frac{f''\dot{R}}{f'}\right]\sigma \quad (19)$$

The above equation is very beneficial in describing the evolution of shear with reference to time. Considering the trace of equation (2) and using the conservation equation satisfied by the energy density of matter (17) in the resulting trace equation, one obtains

$$\dot{R} = \kappa^2 \frac{\theta \rho_m}{f''R - f'} = -\theta \frac{f'R - 2f}{f''R - f'} \quad (20)$$

Using equation (20) in the generalised Friedmann field equation (14), one can obtain

$$\theta^2 = \frac{6f'\sigma^2 + 6\kappa^2(\rho_m + \rho_r) + 3(f'R - f)}{2f'\xi^2} \quad (21)$$

where

$$\xi = 1 + \frac{3}{2} \frac{f''\dot{R}}{f'\theta} = 1 - \frac{3}{2} \frac{f''(f'R - 2f)}{f'(f''R - 2f')} \quad (22)$$

To perform the stability as well as the phase space analysis, it is essential to construct a system of first order differential equations of the cosmological equations using the DSA technique. In this approach, the analysis begins with the introduction of appropriate dimensionless variables. We define the following suitable variables:

$$\Sigma = \frac{\sqrt{3}\sigma}{\xi\theta}, \quad \Omega_r = \frac{3\kappa^2\rho_r}{f'\xi^2\theta^2}, \quad \Omega_m = \frac{3\kappa^2\rho_m}{f'\xi^2\theta^2}, \quad x = \frac{3}{2} \frac{(f'R - f)}{f'\xi^2\theta^2} \quad (23)$$

The above variables are constrained by the modified Friedmann field equation (21) as

$$1 = \Sigma^2 + \Omega_r + \Omega_m + x \quad (24)$$

Using the variables (23) along with the constraint equation (24), one can write the following evolution equations corresponding to the variables Σ , Ω_r and x as:

$$\begin{aligned} \frac{d\Sigma}{d\tau} &= \frac{\Sigma}{2} [-3 + 3\Sigma^2 + \Omega_r - 3x - 9C(R)x + 9D(R)(\Sigma^2 - 1)] \\ \frac{d\Omega_r}{d\tau} &= \Omega_r [-1 + 3\Sigma^2 + \Omega_r - 3x - 9C(R)x + 9D(R)\Sigma^2] \\ \frac{dx}{d\tau} &= x [3 + 3\Sigma^2 + \Omega_r - 3x + 9C(R)(1 - x) + 9D(R)\Sigma^2] \end{aligned} \quad (25)$$

where $\tau \equiv \ln a$

The expression for $C(R)$ and $D(R)$ are given as

$$C(R) = \frac{1}{3\theta} \frac{R\dot{f}'}{(f'R - f)} = -\frac{1}{3} \frac{(f'R - 2f)f''R}{(f'R - f)(f''R - f')} \quad (26)$$

$$D(R) = \frac{1}{3\theta} \frac{f''\dot{R}}{f'} = -\frac{1}{3} \frac{(f'R - 2f)f''}{(f''R - f')f'} \quad (27)$$

In the dynamical system (25), it is realized that the only time (or τ) dependence is contained in the parameters $C(R)$ and $D(R)$. In our analysis, we have considered the condition $R \gg \mu^2$ as the Hu-Sawicki model is found to be practically indistinguishable in this regime [11–14]. The condition $R \ll \mu^2$ is avoided as the correction to GR in this regime is found to be negligible [14]. In the regime $R \gg \mu^2$ corresponding to the Hu-Sawicki model, one can find that

$$C(R) = -\frac{2}{3} \frac{\mu^2}{R} \quad (28)$$

$$D(R) = -\frac{2}{3} c_1 \left(\frac{\mu^2}{R} \right)^2 \quad (29)$$

Therefore in the regime $R \gg \mu^2$, $C(R) \ll 1$ and $D(R) \ll 1$ and these conditions are in general valid for all the heteroclinic orbits in the dynamical system.

As $C(R) \ll 1$ and $D(R) \ll 1$, they can be neglected in the evolution equations (25) as compared to the other terms. As a result the dynamical system (25) does not depend on time (or τ) explicitly, which proves that the

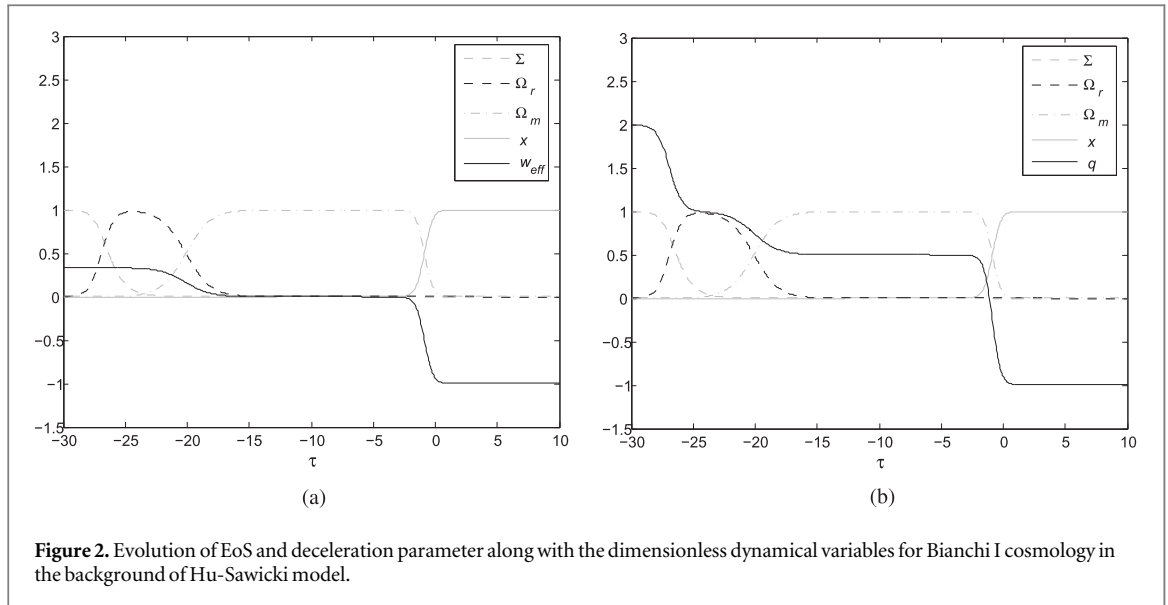


Figure 2. Evolution of EoS and deceleration parameter along with the dimensionless dynamical variables for Bianchi I cosmology in the background of Hu-Sawicki model.

system we have obtained in case of Bianchi I cosmology is an autonomous system. Authors of [54] have provided a detailed investigation on the autonomous dynamical system in case of $f(R)$ gravity.

4.1. The fixed points and exact solutions:

It is important to find the fixed points to study the cosmological behaviour of the dynamical system. The fixed points which can be used to procure the exact solution for the anisotropic Bianchi models are obtained by setting equation (25) equal to zero. In case of anisotropic Bianchi type I model, we encounter anisotropic as well as isotropic fixed points, given by

$$P_s^\pm: (\Sigma, \Omega_r, x) = (\pm 1, 0, 0)$$

$$P_r: (\Sigma, \Omega_r, x) = (0, 1, 0)$$

$$P_m: (\Sigma, \Omega_r, x) = (0, 0, 0)$$

$$P_d: (\Sigma, \Omega_r, x) = (0, 0, 1)$$

Next, we proceed to describe the equation of state (EoS) parameter w_{eff} and the deceleration parameter q in terms of the dynamical variables as

$$w_{eff} = \frac{P_{eff}}{\rho_{eff}} = \frac{\Sigma^2 + \frac{1}{3}\Omega_r - (1 + 3C(R))x + 3D(R)(1 + \Sigma^2) - \Sigma^2\xi^2 + 2\frac{\dot{\xi}}{\xi\theta}}{1 - \Sigma^2\xi^2} \quad (30)$$

$$q = -1 + \frac{3}{2} \left[1 + \Sigma^2 + \frac{1}{3}\Omega_r - (1 + 3C(R))x + 3D(R)(\Sigma^2 + 1) + 2\frac{\dot{\xi}}{\xi\theta} \right] \quad (31)$$

Inserting the values of each variable in equations (30) and (31) and considering $C(R) \ll 1$, $D(R) \ll 1$ and $\xi \approx 1$, one can find the EoS parameter w_{eff} and deceleration parameter q for each fixed point. The evolution of EoS parameter and deceleration parameter along with the dimensionless dynamical variables are plotted in figure 2. These figures show that the Universe starts from the anisotropic state ($w_{eff} = 1/3$, $q = 2$) that isotropizes with the standard radiation ($w_{eff} = 1/3$, $q = 1$) and matter dominated phase ($w_{eff} = 0$, $q = 1/2$) and finally stabilises to the late times de Sitter expansion epoch ($w_{eff} = -1$, $q = -1$).

The relation between the deceleration parameter q , and the volume expansion scalar θ , is written as

$$\frac{\dot{\theta}}{\theta^2} = -\frac{1}{3}(1 + q) \quad (32)$$

Integrating equation (32) for the fixed points corresponding to $q \neq -1$ and setting the big bang time $t_0 = 0$, we obtain

$$\theta = \frac{3}{(1 + q)t} = \frac{3\zeta}{t} \quad (33)$$

We again integrate the above equation in order to find the solution of the average scale factor and the solution is found to be

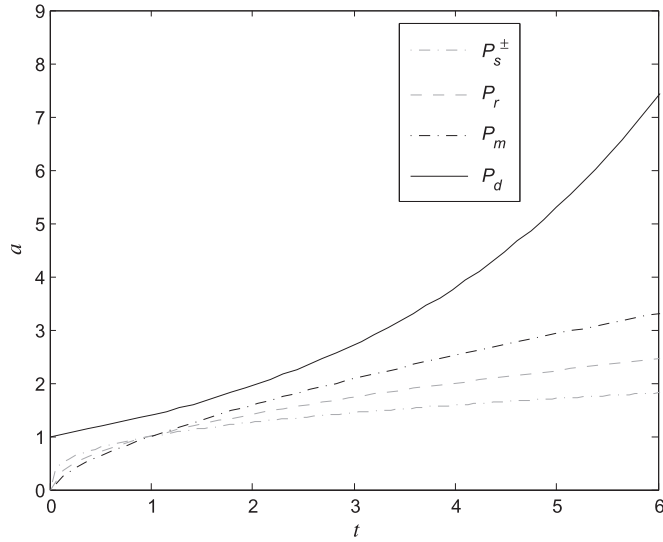


Figure 3. Evolution of scale factor (a) corresponding to Bianchi I cosmology in the background of Hu-Sawicki model for $a_0 = \theta_0 = 1$.

Table 1. Fixed points and their solutions for EoS parameter, deceleration parameter, average scale factor, shear and physical behaviour for Bianchi I cosmology in the background of Hu-Sawicki model.

Points	Ω_m	EoS parameter (w_{eff})	Deceleration parameter (q)	Average scale factor(a)	Shear (σ)	Physical behaviour
P_s^\pm	0	$\frac{1}{3}$	2	$a_0 t ^{\frac{1}{3}}$	$\sigma_0 a_0^{-3} t ^{-1}$	Decelerated expansion
P_r	0	$\frac{1}{3}$	1	$a_0 t ^{\frac{1}{2}}$	0	Decelerated expansion
P_m	1	0	$\frac{1}{2}$	$a_0 t ^{\frac{2}{3}}$	0	Decelerated expansion
P_d	0	-1	-1	$a_0 t ^{\frac{1}{3}\theta_0 t}$	0	Accelerated expansion

$$a = a_0|t|^\zeta \quad (34)$$

where a_0 is a constant and $\zeta = (1 + q)^{-1}$

For the fixed points having $q = -1$, it is found that $\dot{\theta} = 0$. Solutions of such fixed points corresponds to de Sitter solutions which are written as

$$a = a_0 e^{\frac{1}{3}\theta_0 t} \quad (35)$$

Therefore, we find that power law solution exists for $q \neq -1$, whereas de Sitter solution exists for $q = -1$ and these solutions are plotted in figure 3. The average scale factor a , along with the EoS parameter w_{eff} , deceleration parameter q and physical behaviour for the fixed points are summarised in table 1.

4.2. Evolution of shear:

In the following, we study the evolution of shear for the anisotropic fixed points. Such an evolution can be studied from the trace free Gauss-Codazzi equation (19). Equation (19) can be rewritten in terms of dynamical variables (23) as follows

$$\dot{\sigma} = -\frac{1}{\sqrt{3}}[1 + 3D(R)\xi\Sigma]\theta^2 \quad (36)$$

For the anisotropic fixed points P_s^\pm , integration of the above equation leads to

$$\sigma = \sigma_0 a^{-3} = \sigma_0 a_0^{-3}|t|^{-1} \quad (37)$$

The above equation indicates that the shear for the anisotropic fixed points evolves inversely with time and it is shown in figure 4.

4.3. Stability analysis of the fixed points:

In this section, we determine the eigenvalues of the system of equation (25), which are essential in studying the stability. The eigenvalues obtained by linearising the corresponding system of equations are as follows:

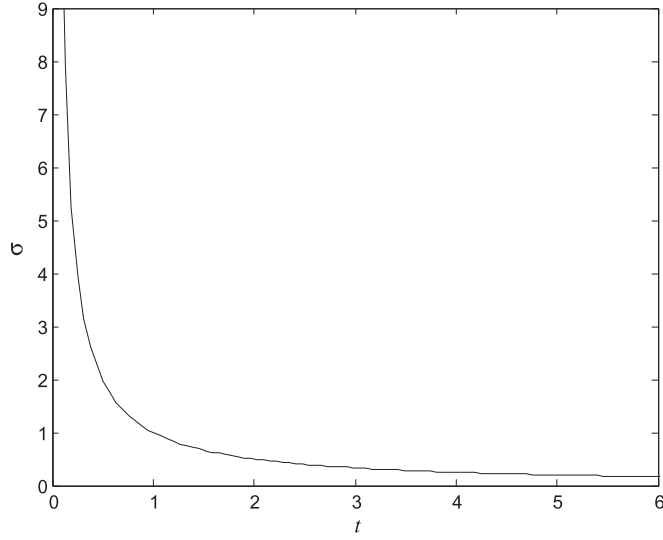


Figure 4. Evolution of shear (σ) corresponding to Bianchi I cosmology in the background of Hu-Sawicki model for $\sigma_0 = 1$.

Table 2. Eigenvalues and stabilities of the fixed points for Bianchi I cosmology in the background of Hu-Sawicki model.

Points	Eigenvalues [$\lambda_1, \lambda_2, \lambda_3$]	Stability
P_s^\pm	[2, 3, 6]	Unstable
P_r	[-1, 1, 4]	Saddle
P_m	$[-\frac{3}{2}, -1, 3]$	Saddle
P_d	[-4, -3, -3]	Stable

$$P_s^\pm: [\lambda_1, \lambda_2, \lambda_3] = [2 + 9D(R), 3 + 9D(R), 6 + 9C(R) + 9D(R)]$$

$$P_r: [\lambda_1, \lambda_2, \lambda_3] = \left[1, 4 + 9C(R), -1 - \frac{9}{2}D(R) \right]$$

$$P_m: [\lambda_1, \lambda_2, \lambda_3] = \left[-1, 3 + 9C(R), -\frac{3}{2} - \frac{9}{2}D(R) \right]$$

$$P_d: [\lambda_1, \lambda_2, \lambda_3] = \left[-3 - 9C(R), -4 - 9C(R), -3 - \frac{9}{2}C(R) - \frac{9}{2}9D(R) \right]$$

In the regime $R \gg \mu^2$ corresponding to the Hu-Sawicki model, it is found that $C(R) \ll 1$ and $D(R) \ll 1$. Therefore, the eigenvalues of the Bianchi I model reduces to the form as listed in table 2. In the corresponding table we also summarise the stability of the fixed points of the dynamical system. For the anisotropic fixed points P_s^\pm , we have all the three eigenvalues positive. Therefore, they correspond to unstable nodes. The points representing the radiation and matter dominated era given by P_r and P_m , respectively, are found to be saddle, as both have positive as well as negative eigenvalues. Finally, for the fixed point P_d , all the eigenvalues are found to be negative, which describe the stable de sitter expansion of the late universe.

4.4. Phase space analysis:

Let us now further investigate the system using phase space analysis in order to study the dynamical behaviour of the Bianchi I cosmology corresponding to the Hu-Sawicki model. The phase portrait analyses of the system are shown in figure 5. We can realize heteroclinic trajectories of the form

$$P_s^\pm \rightarrow P_r \rightarrow P_m \rightarrow P_d$$

This figure displays the progression of the initial unstable anisotropic phase to stable de Sitter expanding universe following the standard radiation and matter era.

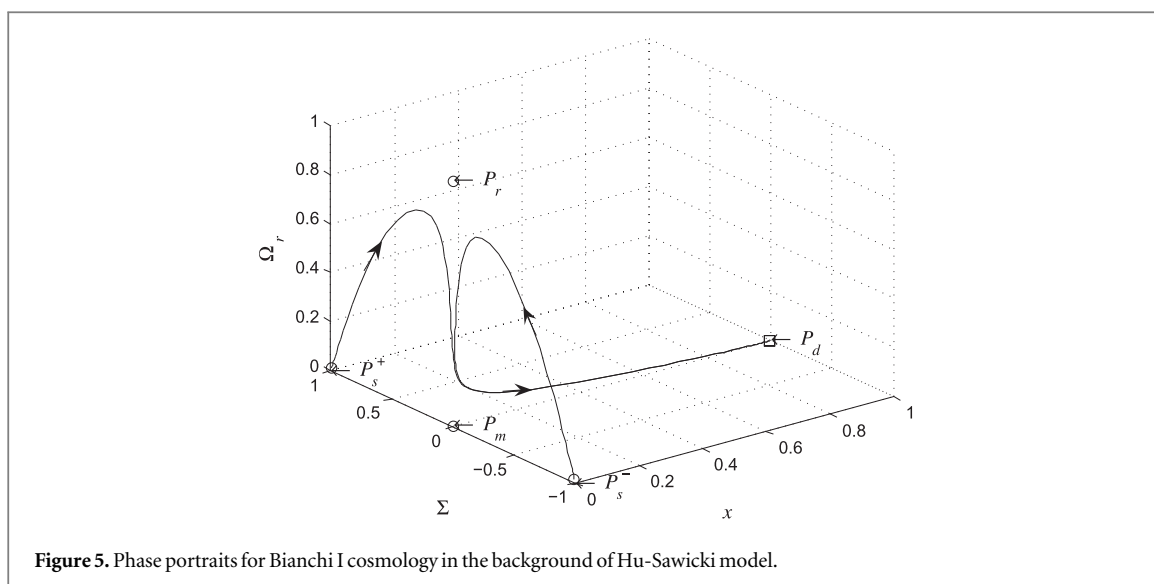


Figure 5. Phase portraits for Bianchi I cosmology in the background of Hu-Sawicki model.

5. Bianchi V metric in Palatini $f(R)$ gravity:

In the following, we study the more complicated form of anisotropic models known as Bianchi type V. The line element of this anisotropic model is defined as

$$ds^2 = -dt^2 + A^2(t)dx^2 + e^{-2px}[B^2(t)dy^2 + C^2(t)dz^2] \quad (38)$$

where $A(t)$, $B(t)$ and $C(t)$ are considered as the expansion scale factors. x, y, z represent the comoving coordinates whereas t denotes the cosmological time and p is a constant.

For this anisotropic model, the field equation (2) modifies to the following forms:

$$f' \left[\frac{3}{2} \frac{\ddot{f}'}{f'} - \frac{3}{2} \left(\frac{\dot{f}'}{f'} \right)^2 + \frac{1}{2} \frac{\dot{f}'}{f'} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} \right] - \frac{1}{2} f = -\kappa \rho \quad (39)$$

$$f' \left[\frac{1}{2} \frac{\ddot{f}'}{f'} + \frac{1}{2} \frac{\dot{f}'}{f'} \left(3 \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\ddot{A}}{A} - \frac{2p^2}{A^2} \right] - \frac{1}{2} f = \kappa P \quad (40)$$

$$f' \left[\frac{1}{2} \frac{\ddot{f}'}{f'} + \frac{1}{2} \frac{\dot{f}'}{f'} \left(\frac{\dot{A}}{A} + 3 \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\ddot{B}}{B} - \frac{2p^2}{A^2} \right] - \frac{1}{2} f = \kappa P \quad (41)$$

$$f' \left[\frac{1}{2} \frac{\ddot{f}'}{f'} + \frac{1}{2} \frac{\dot{f}'}{f'} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + 3 \frac{\dot{C}}{C} \right) + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} + \frac{\ddot{C}}{C} - \frac{2p^2}{A^2} \right] - \frac{1}{2} f = \kappa P \quad (42)$$

$$f' \left[2 \frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right] p = 0 \quad (43)$$

Equation (43) leads to the following equation

$$2 \frac{\dot{A}}{A} = \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \quad (44)$$

Solution of the above equation takes the form

$$A^2 = cBC \quad (45)$$

where c is an integration constant.

The generalised Friedmann field equation in case of the anisotropic Bianchi type V model can be written as

$$\frac{2}{3} f' \left[\theta + \frac{3}{2} \frac{\dot{f}'}{f'} \right]^2 - 2f' \sigma^2 - 6p^2 f' ({}^2K) - f = \kappa(\rho_m + 2\rho_r) \quad (46)$$

where θ, σ and 2K stands for the volume expansion scalar, shear scalar and Gauss curvature of the 3-spheres, respectively, are defined as

$$\theta = 3 \frac{\dot{a}}{a} = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} = 3 \frac{\dot{A}}{A} = \frac{3}{2} \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \quad (47)$$

$$\sigma^2 = \frac{1}{3} \left[\left(\frac{\dot{A}}{A} \right)^2 + \left(\frac{\dot{B}}{B} \right)^2 + \left(\frac{\dot{C}}{C} \right)^2 - \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}\dot{C}}{BC} - \frac{\dot{C}\dot{A}}{CA} \right] = \left(\frac{\dot{A}}{A} - \frac{\dot{C}}{C} \right)^2 \quad (48)$$

$${}^2K = \frac{1}{A^2} \quad (49)$$

The evolution equations for shear and Gauss curvature for the anisotropic Bianchi V space-time are given by

$$\dot{\sigma} = - \left[\theta + \frac{f''\dot{R}}{f'} \right] \sigma \quad (50)$$

$${}^2\dot{K} = -\frac{2}{3}\theta({}^2K) \quad (51)$$

The above equations are very useful in describing the evolution of shear and Gauss curvature of the 3-spheres with reference to time.

Using equation (20) in the generalised Friedmann field equation (46), one can obtain

$$\theta^2 = \frac{6f'\sigma^2 + 6\kappa(\rho_m + \rho_r) + 18p^2f'({}^2K) + 3(f'R - f)}{2f'\xi^2} \quad (52)$$

In addition to the variables considered in the previous Bianchi I model, here we have considered one extra dimensionless variable representing the Gauss curvature, given by

$$K = -\frac{3p^2({}^2K)}{\xi^2\theta^2} \quad (53)$$

Therefore, the variables (23) and (53) are constrained by the modified Friedmann field equation (52) as

$$1 = \Sigma^2 + \Omega_r + \Omega_m - 3K + x \quad (54)$$

Using the variables (23) and (53) along with the constraint equation (54), one can write the following evolution equations corresponding to the variables Σ , Ω_r , K and x as:

$$\begin{aligned} \frac{d\Sigma}{d\tau} &= \frac{\Sigma}{2} [-3 + 3\Sigma^2 + \Omega_r + 3K - 3x - 9C(R)x + 9D(R)(\Sigma^2 + 3K - 1)] \\ \frac{d\Omega_r}{d\tau} &= \Omega_r [-1 + 3\Sigma^2 + \Omega_r + 3K - 3x - 9C(R)x + 9D(R)(\Sigma^2 + 3K)] \\ \frac{dK}{d\tau} &= K [1 + 3\Sigma^2 + \Omega_r + 3K - 3x - 9C(R)x + 9D(R)(\Sigma^2 + 3K + 1)] \\ \frac{dx}{d\tau} &= x [3 + 3\Sigma^2 + \Omega_r + 3K - 3x + 9C(R)(1 - x) + 9D(R)(\Sigma^2 + 3K)] \end{aligned} \quad (55)$$

In the dynamical system (55), it is noticed that only time (or τ) dependence is contained in the parameter $C(R)$ and $D(R)$. In the regime $R \gg \mu^2$ corresponding to the Hu-Sawicki model, one can find that $C(R) \ll 1$ and $D(R) \ll 1$ and hence are neglected in the analysis. Therefore the dynamical system (55) does not depend on time (or τ) explicitly, which proves that the system we have obtained in case of Bianchi V cosmology is an autonomous system.

5.1. The fixed points and exact solutions:

The fixed points that are used to procure the exact solutions for the anisotropic Bianchi type V model are obtained by setting (55) equal to zero. In case of this anisotropic model we encounter anisotropic as well as isotropic fixed points, given by

$$P_s^\pm: (\Sigma, \Omega_r, K, x) = (\pm 1, 0, 0, 0)$$

$$P_r: (\Sigma, \Omega_r, K, x) = (0, 1, 0, 0)$$

$$P_m: (\Sigma, \Omega_r, K, x) = (0, 0, 0, 0)$$

$$P_k: (\Sigma, \Omega_r, K, x) = \left(0, 0, -\frac{1}{3}, 0 \right)$$

$$P_d: (\Sigma, \Omega_r, K, x) = (0, 0, 0, 1)$$

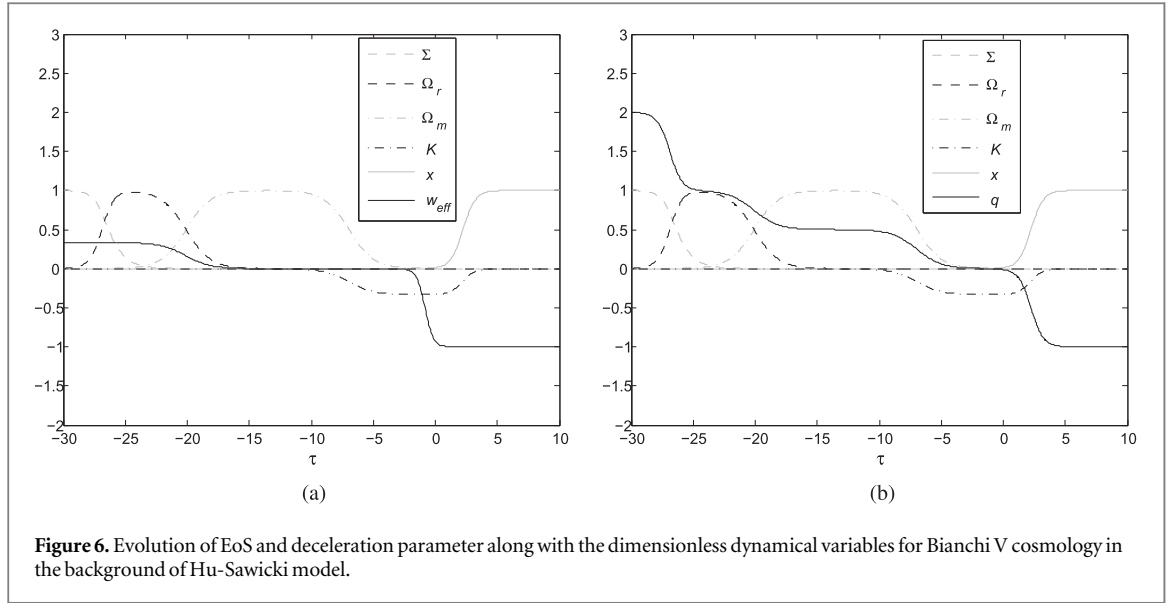


Figure 6. Evolution of EoS and deceleration parameter along with the dimensionless dynamical variables for Bianchi V cosmology in the background of Hu-Sawicki model.

Table 3. Fixed points and their solutions for EoS parameter, deceleration parameter, average scale factor, shear, Gauss curvature and physical behaviour for Bianchi V cosmology in the background of Hu-Sawicki model.

Points	Ω_m	EoS parameter (w_{eff})	Deceleration parameter (q)	Average Scale factor(a)	Shear (σ)	Gauss curvature (2K)	Physical behaviour
P_s^\pm	0	$\frac{1}{3}$	2	$a_0 t ^{\frac{1}{3}}$	$\sigma_0 a_0^{-3} t ^{-1}$	0	Decelerated expansion
P_r	0	$\frac{1}{3}$	1	$a_0 t ^{\frac{1}{2}}$	0	0	Decelerated expansion
P_m	1	0	$\frac{1}{2}$	$a_0 t ^{\frac{2}{3}}$	0	0	Decelerated expansion
P_k	0	$-\frac{1}{3}$	0	$a_0 t $	0	${}^2K_0 a_0 t ^{-2}$	Decelerated expansion
P_d	0	-1	-1	$a_0 t ^{\frac{1}{3}\theta_0 t}$	0	0	Accelerated expansion

The equation of state (EoS) parameter w_{eff} and the deceleration parameter q in terms of the dynamical variables in case of Bianchi V space-time are given by

$$w_{eff} = \frac{P_{eff}}{\rho_{eff}} = \frac{1}{1 - \xi^2(\Sigma^2 + 3K)} \left[\Sigma^2 + \frac{\Omega_r}{3} + K - (1 + 3C(R))x + 3D(R)(1 + \Sigma^2 + 3K) - \xi^2(\Sigma^2 + K) + 2\frac{\dot{\xi}}{\xi\theta} \right] \quad (56)$$

$$q = -1 + \frac{3}{2} \left[1 + \Sigma^2 + \frac{1}{3}\Omega_r + K - (1 + 3C(R))x + 3D(R)(\Sigma^2 + 3K + 1) + 2\frac{\dot{\xi}}{\xi\theta} \right] \quad (57)$$

Considering $C(R) \ll 1, D(R) \ll 1, \xi \approx 1$ and substituting the value of each variable in equations (56) and (57), one can obtain the EoS parameter w_{eff} and deceleration parameter q for the associated fixed points corresponding to Bianchi V model. In figure 6, we plot the evolution of EoS parameter and deceleration parameter along with the dimensionless dynamical variables for this anisotropic model. These figures depict that the Universe evolves from the anisotropic phase ($w_{eff} = 1/3, q = 2$) to the isotropic radiation dominated ($w_{eff} = 1/3, q = 1$), matter dominated ($w_{eff} = 0, q = 1/2$) and spatially non-flat ($w_{eff} = -1/3, q = 0$) phase, respectively, and enters the stable phase of late times de Sitter expansion ($w_{eff} = -1, q = -1$).

The EoS parameter w_{eff} and deceleration parameter q along with average scale factor a and physical behaviour are summarised in table 3. Table 3 shows that the points P_s^\pm, P_r and P_m admit power law solution whereas P_k and P_d attain Milne and de Sitter solution respectively. These solutions are plotted in figure 7.

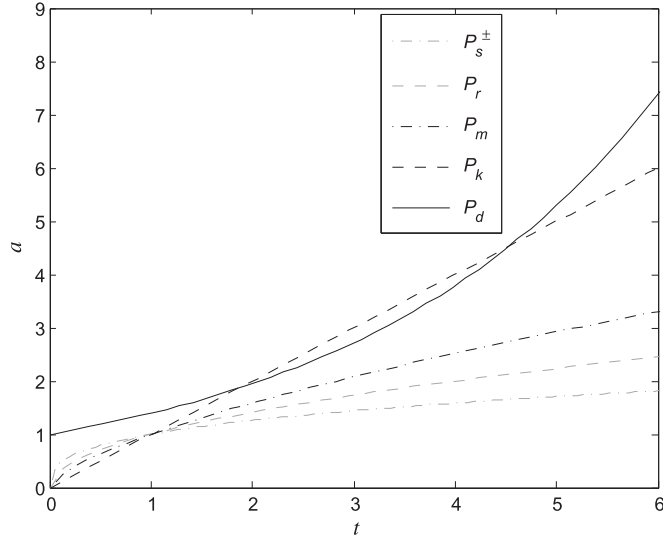


Figure 7. Evolution of scale factor (a) corresponding to Bianchi V cosmology in the background of Hu-Sawicki model for $a_0 = \theta_0 = 1$.

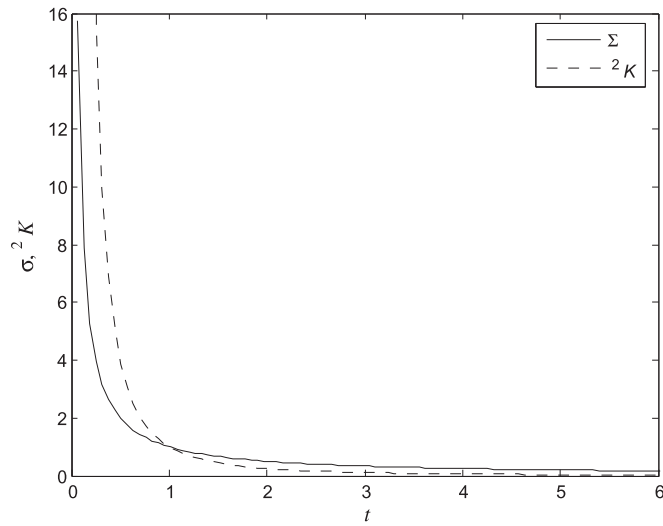


Figure 8. Evolution of shear (σ), Gauss curvature (2K) corresponding to Bianchi V cosmology in the background of Hu-Sawicki model for $\sigma_0 = 1$, ${}^2K_0 = 1$ and $a_0 = 1$.

5.2. Evolution of Shear:

Now we analyse the trace free Gauss-Codazzi equation (50), in order to find the evolution of shear for the anisotropic fixed points (P_s^\pm). The Gauss-Codazzi equation (50), in terms of dynamical variables (23), (53) for Bianchi V space-time is given by

$$\dot{\sigma} = -\frac{1}{\sqrt{3}}[(1 + 3D(R))\xi\Sigma]\theta^2 \quad (58)$$

Integrating the above equation we obtain

$$\sigma = \sigma_0 a^{-3} = \sigma_0 a_0^{-3} |t|^{-1} \quad (59)$$

The above solution implies that the shear corresponding to each anisotropic fixed point diminishes with the evolution of cosmic time and this is displayed in figure 8.

5.3. Evolution of Gauss curvature of the 3-spheres:

The evolution of Gauss curvature of the 3-spheres for the spatially non-flat fixed point (P_k) can be explored from equation (51). Integration of the equation (51) gives

Table 4. Eigenvalues and stabilities of the fixed points for Bianchi V cosmology in the background of Hu- Sawicki model.

Points	Eigenvalues [$\lambda_1, \lambda_2, \lambda_3, \lambda_4$]	Stability
P_s^\pm	[2,3,4,6]	Unstable
P_r	[-1, 1, 2, 4]	Saddle
P_m	$[-\frac{3}{2}, -1, 1, 3]$	Saddle
P_k	[-2, -2, -1, 2]	Saddle
P_d	[-4, -3, -3, -2]	Stable

$${}^2K = {}^2K_0 a^{-2} = {}^2K_0 a_0^{-2} |t|^{-2} \quad (60)$$

The above equation shows that the Gauss curvature dies down as time progresses and this is shown in figure 8.

5.4. Stability analysis of the fixed points:

In this section, we analyze the eigenvalues of the system of equation (55), in order to study the stability corresponding to the Bianchi V cosmology. The eigenvalues obtained by linearising the system of equation (55) are as follows:

$$P_s^\pm: [\lambda_1, \lambda_2, \lambda_3, \lambda_4] = [2 + 9D(R), 3 + 9D(R), 4 + 18D(R), 6 + 9C(R) + 9D(R)]$$

$$P_r: [\lambda_1, \lambda_2, \lambda_3, \lambda_4] = \left[1, 4 + 9C(R), 2 + 9D(R), -1 - \frac{9}{2}D(R) \right]$$

$$P_m: [\lambda_1, \lambda_2, \lambda_3, \lambda_4] = \left[-1, 3 + 9C(R), 1 + 9D(R), -\frac{3}{2} - \frac{9}{2}D(R) \right]$$

$$P_k: [\lambda_1, \lambda_2, \lambda_3, \lambda_4] = [-1 - 9D(R), -2 - 9D(R), -2 - 9D(R), 2 + 9C(R) - 9D(R)]$$

$$P_d: [\lambda_1, \lambda_2, \lambda_3, \lambda_4] = \left[-3 - 9C(R), -4 - 9C(R), -2 - 9C(R) - 9D(R), -3 - \frac{9}{2}C(R) - \frac{9}{2}9D(R) \right]$$

Considering $C(R) \ll 1$ and $D(R) \ll 1$, the eigenvalues regarding the Bianchi V cosmology are modified to the form as summarised in table 4. In this table, we also list the stability of the fixed points corresponding to the dynamical system. For the anisotropic fixed points P_s^\pm , the linearized matrix contains four positive eigenvalues. Hence, these fixed points correspond to unstable nodes. The points denoting the radiation dominated, matter dominated and spatially non-flat universe given by P_r, P_m and P_k respectively, are found to be saddle, as the linearized matrix have positive as well as negative eigenvalues. Finally, for the isotropic de Sitter fixed point P_d , all the eigenvalues are found to be negative, describing the stable late time accelerated expansion of the Universe.

5.5. Phase space analysis:

This section is devoted for the investigation of the system using phase space analysis in order to study the dynamical behaviour of the Bianchi V cosmology corresponding to the Hu-Sawicki $f(R)$ model. The phase portrait analyses of the system for this model are displayed in figure 9. In figure 9(a) we observe the heteroclinic trajectories of the form

$$P_s^\pm \rightarrow P_m \rightarrow P_k \rightarrow P_d$$

This figure is plotted for $\Omega_r = 0$, which shows the progression of the unstable anisotropic phase to the stable isotropic de Sitter expanding scenario following by the matter dominated and spatially non-flat isotropic age.

Figure 9(b) shows the heteroclinic trajectories for $\Sigma = 0$, that gives

$$P_r \rightarrow P_m \rightarrow P_k \rightarrow P_d$$

This figure displays the evolution of the radiation dominated phase to the matter dominated, than to spatially non-flat and finally to the accelerated expanding period. Figure 9(c) displays the heteroclinic trajectories for $K = 0$, that gives

$$P_s^\pm \rightarrow P_r \rightarrow P_m \rightarrow P_d$$

This figure depicts the progression of the unstable anisotropic universe to the stable expanding de Sitter phase following the radiation and matter dominated eras.

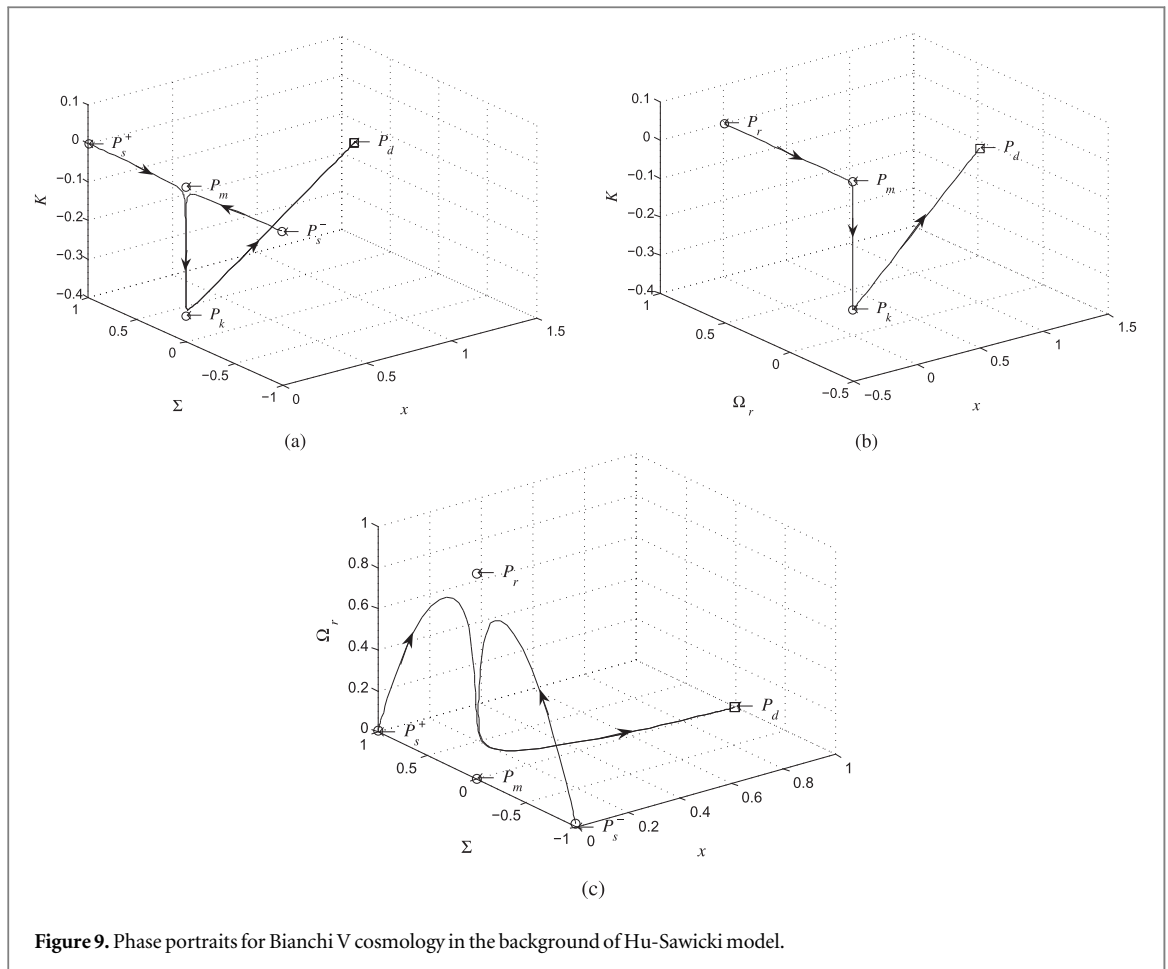


Figure 9. Phase portraits for Bianchi V cosmology in the background of Hu-Sawicki model.

6. Conclusion:

Field equations and the fixed points for Palatini $f(R)$ gravity in the background of anisotropic Bianchi I and V have been derived employing the dynamical system technique. We present the solutions associated with each fixed point considering an exclusive form of $f(R)$ initially proposed by Hu and Sawicki.

Our work begins with one of the simplest type of anisotropic model known as Bianchi type I. In this model, we have identified two anisotropic (P_s^\pm) and three isotropic (P_r, P_m, P_d) fixed points. Stable de Sitter solution exists for the isotropic fixed point describing the late time accelerating phase of the Universe. Points indicating the radiation and matter dominated epochs behave as saddles whereas the anisotropic fixed points describing the shear are found to be unstable. Thus, in this model, the Universe starts from the anisotropic scenario that isotropizes with the standard radiation and matter dominated era and attain the late time de Sitter accelerating epoch.

The work is further extended to another form of anisotropic model known as Bianchi type V. In addition to the points obtained in the previous model, this model gives rise to an additional isotropic fixed point (P_k) describing a non-flat universe. Thus, in this case, we have two anisotropic (P_s^\pm) and four isotropic (P_r, P_m, P_k, P_d) fixed points. The isotropic de Sitter point describing the acceleration phase of the late universe is found to be stable whereas the radiation, matter and spatially non-flat isotropic fixed points behave as saddle points. The point corresponding to anisotropic age is found to be unstable. Therefore, this model is able to accomplish a sequence originating from an anisotropic phase which subsequently ends up in an isotropic de Sitter expanding stage, following a radiation, matter dominated as well as spatially non-flat universe.

Anisotropic models of these types which are very effective in explaining the anisotropy of early universe, depict that the anisotropy dies down as $(\text{time})^{-1}$. The explicit treatment of the anisotropic model Bianchi type V furnishes the Gauss curvature evolution and shows that curvature dissipates as $(\text{time})^{-2}$.

In [5], the authors have been able to obtain the sequence of radiation-dominated, matter-dominated and late time acceleration for FLRW cosmology considering $f(R) = R - \beta/R^n$. Authors of [11] could reproduce the same sequence considering the Hu-Sawicki $f(R)$ gravity in Palatini formalism. Working in the background of anisotropic Bianchi I and V model, we have been able to regenerate the same sequence. Additionally, we have been able to obtain a sequence that starts from anisotropic universe which ends into a stable accelerating phase.

The authors of [28–30, 39, 44–47] have shown that the anisotropic universe accomplished the isotropic stage with the evolution of time in case of anisotropic Bianchi models, which is consistent with our analysis.

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